Applied Mathematics

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2.5.6 (The Leaky bucket) The following example (Hubbard and West 1991, p.159) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it can you figure out when the bucket was full ? NO, of course not! It could have finished emptying a minute age, ten minutes ago, or whatever. The solution to the corresponding differental equation must be non-unique when intergrated backwards in time.

Here's a crude model of the situation. Let $h(t)$ = height of the water remaining in the bucket at time $t; a$ = area of the hole; A=cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

a) Show that $av(t) = A\dot{h}(t)$. What physical law are you invoking?

The rate at which the water in the bucket decreases $A\dot{h}$ must equal the rate at which the water leaves the bucket $av(t)$ thus $av(t) = A\dot{h}$.

The Physical law being invoked here is the conservation of mass.

b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh and that the water has density p. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation v^2gh.

The potential energy of the water in the bucket is given by

$$V = \int_0 hgApydy = \frac{1}{2}gAph^2$$

We also know that $\dot{K} = \frac{1}{2}pA\dot{h}v^2$ is the rate at which the kinetic energy is transported out of the bucket by the water.

Next we take our transported kinetic energy and set it equal to our potential energy

$$\frac{1}{2}gAph^2 = \frac{1}{2}pAh\dot{v}^2$$
$$v^2 = 2gh$$

c) Combining (a) and (b), show $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}(\frac{a}{A})$.

First solve for v we get,

$$v = \sqrt{2gh}$$

in then we plug v into our equation found in part a.

$$a\sqrt{2gh} = A\dot{h}$$

. then solve for $\dot{h}$

$$\dot{h} = \frac{a\sqrt{2gh}}{A}$$

rearranging we get

$$\sqrt{h}\sqrt{2g}\frac{a}{A} = C\sqrt{h} = \dot{h}$$

d) $h(0) = 0$ (bucket empty at $t = 0$, show that the solution for $h(t)$ is non-unique *in backwards time*, i.e., for $t < 0$. We can find the following solutions, that are valid for $t < 0$

$$h(t) = \begin{cases} 0 & \text{for } t_0 \leq t \leq 0, \\ (\frac{C}{2}(t_0 - t))^2 & \text{for } t \leq t_0. \end{cases}$$